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Eigenvalue Integro-Differential Equations for Orthogonal Polynomials on the Real Line*

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Abstract

The one-dimensional harmonic oscillator wave functions are solutions to a Sturm-Liouville problem posed on the whole real line. This problem generates the Hermite polynomials. However, no other set of orthogonal polynomials can be obtained from a Sturm-Liouville problem on the whole real line. In this paper we show how to characterize an arbitrary set of polynomials orthogonal on $(-\infty, \infty)$ in terms of a system of integro-differential equations of Hartree-Fock type. This system replaces and generalizes the linear differential equation associated with a Sturm-Liouville problem. We demonstrate our results for the special case of Hahn-Meixner polynomials.

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1 Introduction

In this paper we will examine the system of N coupled nonlinear integro-differential equations defined on the whole real line

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}-\frac{1}{4}U''(x)+\frac{1}{8}U'^2(x)+\mu_i^{(N)}\right]u_i(x)-\frac{1}{2}\sum_{j=0}^{N-1}\int_{-\infty}^{\infty}dy\,u_j^*(y)$$
$$\times\frac{U'(x)-U'(y)}{x-y}\left[u_i(x)u_j(y)-u_j(x)u_i(y)\right]\ =0\quad(i=0,\,1,\,2,\,\ldots,\,N-1).\qquad(1)$$

This system is an eigenvalue problem in which there are N eigenfunctions $u_i(x)$ and N corresponding eigenvalues $\mu_i^{(N)}$. The eigenfunctions are constrained by the orthonormality condition

$$\int_{-\infty}^{\infty}dx\,u_i(x)u_j(x)=\delta_{ij}.\qquad(2)$$

The single function $U(x)$ together with the integer N completely characterize the eigenvalue problem under consideration. The function $U(x)$ is restricted by the requirements that it be twice continuously differentiable, that it be bounded from below, and that it satisfy the inequality constraint

$$\int_{-\infty}^{\infty}dx\,e^{-U(x)}<\infty.\qquad(3)$$

This eigenvalue problem constitutes a system of Hartree-Fock equations that describe the ground state of a one-dimensional Fermi gas having N particles. The functions $u_i(x)$ are the single-particle states of the Fermi gas and the numbers $\mu_i^{(N)}$ are the corresponding chemical potentials associated with (2).

As will be shown later, (1-3) arise in the study of random Hermitian N-dimensional matrices Φ whose statistical weight is given by [1]

$$d\mu(\Phi)=\frac{1}{Z}e^{-\mathrm{Tr}\,U(\Phi)}d^{N^2}\Phi,\qquad(4)$$

where Z is a normalization factor [2]. Associated with such an ensemble of random matrices is a set of polynomials $\mathcal{P}_i(x)$ that are orthonormal on the whole real line with respect to

the weight in (4) [3]:

$$\int_{-\infty}^{\infty} dx\, \mathcal{P}_i(x)\mathcal{P}_j(x)e^{-U(x)} = \delta_{ij}. \tag{5}$$

The relation between the polynomials $\{\mathcal{P}_i(x)\}$ and the eigenfunctions $\{u_i(x)\}$ of (1) is simply

$$u_i(x) = e^{-\frac{1}{2}U(x)}\mathcal{P}_i(x). \tag{6}$$

If we substitute (6) into (1) and use (2) we can calculate the eigenvalues $\{\mu_i^{(N)}\}$. The connection between the eigenvalue problem (1) and the theory of random matrices and orthogonal polynomials will be discussed in more detail in Sec. III. However, the principal consequence of this connection is that (6) is the *unique* solution $[u_0(x),\, u_1(x),\, u_2(x),\, \ldots, u_{N-1}(x)]$ to (1) and (2) for a given function $U(x)$. Physically, this uniqueness is a consequence of the Fermi gas being in its ground state. To the best of our knowledge, the eigenvalue problem in (1) is new in the general theory of orthogonal polynomials.

In the special case $U(x) = \frac{1}{2}x^2$, (1) simplifies to a set of N decoupled linear differential equations:

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}x^2 + 2\mu_i^{(N)} - N + \frac{1}{2}\right) u_i(x) = 0. \tag{7}$$

These are the Schrödinger equations for the lowest N eigenstates of the one-dimensional harmonic oscillator from which we immediately infer that $\mu_i^{(N)} = \frac{1}{2}(N-1-i)$. In the case of the harmonic oscillator, $\mu_i^{(N)}$ is the depth of the ith state below the Fermi level.

We will now show that the Hermite polynomials are the *only* set of orthogonal polynomials that are determined by a Sturm-Liouville problem *on the whole real line*. Let $\{\mathcal{P}_i(x)\}$ be a set of orthogonal polynomials satisfying (5). Assume further that the functions $\{u_i(x)\}$ as defined in (6) are the eigenfunctions of the general Sturm-Liouville problem [4]

$$-\frac{d}{dx}\left[p(x)\frac{d}{dx}u_i(x)\right] + [q(x) - \mu_i W(x)]u_i(x) = 0 \quad (-\infty < x < \infty), \tag{8}$$

where $p(x) > 0$ and $q(x)$ are continuous functions and $\{\mu_i\}$ are the associated eigenvalues. Since the domain of (8) is infinite, the only boundary condition imposed on the eigenfunctions $\{u_i(x)\}$ is the normalization condition (2). We emphasize that $p(x)$ is strictly positive for all x on the whole line.[1] Moreover, since the Sturm-Liouville problem in (8) is one dimensional, the orthonormality conditions (5) imply that the spectrum $\{\mu_i\}$ is discrete, nondegenerate, and bounded from below. Following (2) we note further that the weight in (8) is fixed by $W(x) = 1$. Substituting the functions $\{u_i(x)\}$ from (6) into the eigenvalue problem (8), we obtain

$$p(x)\mathcal{P}_i''(x)+\left[2\frac{w'(x)}{w(x)}p(x)+p'(x)\right]\mathcal{P}_i'(x)+\left[p(x)\frac{w''(x)}{w(x)}+p'(x)\frac{w'(x)}{w(x)}+\mu_i-q(x)\right]\mathcal{P}_i(x)=0, \tag{9}$$

where $w(x) = e^{-\frac{1}{2}U(x)}$. Equation (9) holds for all real x and may be used to express $p(x)$, $q(x)$, $w(x)$, and the spectrum $\{\mu_i\}_{i\geq 3}$ in terms of the first three polynomials and their corresponding eigenvalues as we now show.

Let the first three orthonormal polynomials in (6) be

$$\mathcal{P}_0(x) = 1, \qquad \mathcal{P}_1(x) = ax + b \quad (a \neq 0), \qquad \mathcal{P}_2(x) = cx^2 + dx + e \quad (c \neq 0), \tag{10}$$

where we have assumed the normalization

$$\int_{-\infty}^{\infty} dx\ w^2(x) \ = \ 1. \tag{11}$$

Substituting $\mathcal{P}_0$ and $\mathcal{P}_1$ from (10) into (9) we obtain

$$\begin{aligned} q(x) &= p(x)\frac{w''(x)}{w(x)} + p'(x)\frac{w'(x)}{w(x)} + \mu_0, \\ &\left[2\frac{w'(x)}{w(x)}p(x) + p'(x)\right] + (\mu_1 - \mu_0)\left(x + \frac{b}{a}\right) = 0. \end{aligned} \tag{12}$$

[1]In our case the domain of definition of the Sturm-Liouville problem is the whole real line, $-\infty < x < \infty$. In cases where the domain of definition is a compact subset of the real line, $p(x)$ may vanish at its boundary points, which then become singular points of the Sturm-Liouville problem. In the case considered here the function $p(x)$ may vanish only at $x = \pm\infty$.

These results allow us to rewrite (9) as

$$p(x)\ \mathcal{P}_i''(x) - (\mu_1 - \mu_0)\left(x + \frac{b}{a}\right)\mathcal{P}_i'(x) + (\mu_i - \mu_0)\,\mathcal{P}_i(x) = 0. \tag{13}$$

Substituting $\mathcal{P}_2(x)$ from (10) into (13) we find that

$$\begin{aligned} p(x) &= \left(\mu_1 - \mu_0 - \frac{\mu_2 - \mu_0}{2}\right)x^2 + \left[(\mu_1 - \mu_0)\,\frac{b}{a} - (\mu_2 - \mu_1)\,\frac{d}{2c}\right]x \\ &\quad + \frac{1}{2c}\left[(\mu_1 - \mu_0)\,\frac{bd}{a} - (\mu_2 - \mu_0)\,e\right]. \end{aligned} \tag{14}$$

The function $p(x)$ is strictly positive for any real value of x. Thus, either $p(x)$ is a quadratic polynomial and the requirement of positivity implies the two inequalities

$$\begin{aligned} \mu_2 + \mu_0 - 2\mu_1 &< 0, \\ (\mu_1 - \mu_0)^2\left(\frac{b}{a} - \frac{d}{2c}\right)^2 + (\mu_2 - \mu_0)\,(\mu_2 + \mu_0 - 2\mu_1)\left[\left(\frac{d}{2c}\right)^2 - \frac{e}{c}\right] &< 0, \end{aligned} \tag{15}$$

or else $p(x)$ is a positive constant and we have

$$\begin{aligned} \mu_1 - \mu_0 - \frac{\mu_2 - \mu_0}{2} &= 0, \\ (\mu_1 - \mu_0)\,\frac{b}{a} - (\mu_2 - \mu_1)\,\frac{d}{2c} &= 0, \\ \frac{1}{2c}\left[(\mu_1 - \mu_0)\,\frac{bd}{a} - (\mu_2 - \mu_0)\,e\right] &> 0. \end{aligned} \tag{16}$$

Equation (14) is only one of many possible expressions for $p(x)$. Indeed, one may alternatively solve (13) for $p(x)$ in terms of any one of the polynomials $\{\mathcal{P}_i\}_{i\geq 3}$ as

$$p(x) = \frac{\left(x + \frac{b}{a}\right)(\mu_1 - \mu_0)\,\mathcal{P}_i'(x) - (\mu_i - \mu_0)\,\mathcal{P}_i(x)}{\mathcal{P}_i''(x)}, \tag{17}$$

which owing to (14), must be either a quadratic polynomial in x or a positive constant for all i. This is precisely the condition that determines the whole spectrum in terms of

$\mu_0,\ \mu_1,$ and μ_2 . To see this, we take the limit of (17) as $|x| \to \infty$ and using (14) we find that

$$\mu_i = \left[\frac{\mu_2 - \mu_0}{2} - (\mu_1 - \mu_0)\right] i^2 + \left[2(\mu_1 - \mu_0) - \frac{\mu_2 - \mu_0}{2}\right] i + \mu_0. \tag{18}$$

The condition that the spectrum in (18) increases monotonically implies the inequality

$$\mu_2 + \mu_0 - 2\mu_1 \geq 0. \tag{19}$$

If $p(x)$ is a quadratic polynomial, then the inequality $\mu_2 + \mu_0 - 2\mu_1 > 0$ must hold, but this contradicts the first equation in (15). It also contradicts the second inequality in (15) because if both these relations are valid, they imply the inequality

$$\left(\frac{d}{2c}\right)^2 - \frac{e}{c} < 0. \tag{20}$$

However, from (10) this is precisely the condition that $\mathcal{P}_2(x)$ be nonvanishing on the real line, which implies that its sign never alternates along the real line. This conclusion contradicts the orthogonality of $\mathcal{P}_2$ and $\mathcal{P}_0$.[2] Therefore, (15) can never hold, and (16), which implies that $p(x)$ is a positive constant p, is the only possibility. Finally, using (11-12) we find that

$$w^2(x) = e^{-U(x)} = \left(\frac{2\pi p}{\mu_1 - \mu_0}\right)^{1/2} e^{-\frac{\mu_1 - \mu_0}{2p}\left(x + \frac{b}{a}\right)^2} \tag{21}$$

and

$$q(x) = \frac{(\mu_1 - \mu_0)^2}{4p}\left(x + \frac{b}{a}\right)^2 + \frac{3}{2}\mu_0 - \frac{1}{2}\mu_1, \tag{22}$$

for which (8) becomes the Schrödinger equation for an harmonic oscillator whose equilibrium point is at $x_0 = -b/a$, and the polynomials $\{\mathcal{P}_i(x)\}$ are the corresponding (shifted)

[2] There is another way to see that (20) is false. We note that the coefficient e in $\mathcal{P}_2(x)$ in (10) is actually not arbitrary; it may be determined in terms of the coefficients a, b, c, and d as follows. We consider the following three orthonormality relations: (1) $\int dx\, w^2(x)\mathcal{P}_0(x)\mathcal{P}_1(x) = 0$; (2) $\int dx\, w^2(x)\mathcal{P}_0(x)\mathcal{P}_2(x) = 0$; (3) $\int dx\, w^2(x)\mathcal{P}_1^2(x) = 1$. These equations may be regarded as three linear simultaneous equations for three unknowns, the first and second moments of $w^2(x)$ [the zeroth moment is given in (11)] and the coefficient e. The solution for e is $e = (abd - c - b^2c)/a^2$. When this value is substituted into the inequality in (20), the inequality takes the form $[(ad - 2bc)^2 + 4c^2]/(4a^2c^2) < 0$, which is manifestly impossible.

Hermite polynomials and the eigenvalues are $\mu_i = (\mu_1 - \mu_0)i + \mu_0$. This concludes the demonstration that the Hermite polynomials are the only set of orthogonal polynomials that are determined by a Sturm-Liouville problem on the whole real line.

This paper is organized as follows. In Sec. II we present a proof of the integro-differential eigenvalue equation (1) based on a variational calculation. In Sec. III we give a general formula for the eigenvalues associated with an arbitrary system of polynomials having an even weight function. This formula for the eigenvalues is useful because it gives the spectrum for a one-dimensional system of interacting fermions. Finally, in Sec. IV we illustrate equations (1) and (2) for the special case of Hahn-Meixner polynomials. We calculate the first few eigenvalues and find some of their general features.

2 Orthonormal Polynomials on the Real Line

In this section we use a variational principle to prove that polynomials orthonormal relative to an arbitrary weight $w^2(x) = e^{-U(x)}$ on the whole real line must obey the system of Hartree-Fock equations in (1) [2]. We show that as a consequence the functions in (6) are the *unique* solution of (1). [Recall from Sec. I that the system of equations in (1) reduces to the trivial set of uncoupled Schrödinger equations (7) for the eigenstates of an harmonic oscillator (i.e., the case of Hermite polynomials) when the weight is Gaussian $w^2(x) = e^{-x^2/2}$.]

Consider a quantum mechanical system whose degrees of freedom are elements of an $N \times N$ Hermitian matrix Φ. The Hamiltonian for this system is the positive semi-definite operator given by

$$\mathcal{H} = \frac{1}{2}\mathrm{Tr}\left[\left(-\frac{\partial}{\partial\Phi} + \frac{1}{2}U'(\Phi)\right)\left(\frac{\partial}{\partial\Phi} + \frac{1}{2}U'(\Phi)\right)\right]. \tag{23}$$

We will require that $U(\Phi)$ be a matrix potential function satisfying the restriction in (3). This clearly implies that the potential in (23) is bounded from below and grows to plus infinity as the matrix eigenvalues become infinite. Hence, the Schrödinger operator $\mathcal{H}$ in (23) has a well-defined spectrum. This Hamiltonian is symmetric under the adjoint $U(N)$ transformation $\Phi \to \Omega\Phi\Omega^\dagger$. Therefore, $\mathcal{H}$ possesses a unique ($U(N)$ singlet) normalizable ground-state vector $\Psi_0(\Phi)$. This ground state has energy zero and is given by[3]

$$\begin{aligned} \Psi_0(\Phi) &= \frac{1}{\sqrt{Z}}\exp[-\frac{1}{2}\mathrm{Tr}U(\Phi)] \\ Z &= \int d^{N^2}\Phi\,\exp[-\mathrm{Tr}\,U(\Phi)]. \end{aligned} \tag{24}$$

As is well known, the Laplacian over Hermitian matrices acquires the form [3, 5, 18]

$$-\mathrm{Tr}\frac{\partial^2}{\partial\Phi^2} = -\frac{1}{\Delta(x)}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\Delta(x) + [\mathrm{U}(N) \text{ angular momentum terms}], \tag{25}$$

[3]It is easy to verify that this state has zero energy by substituting (24) into (23). Thus, since $\mathcal{H}$ is non-negative, Ψ_0 in (24) is the ground state because the latter is unique.

where x_i are the matrix eigenvalues and $\Delta(x_i)$ is the Vandermonde determinant.

For a generic potential $U(\Phi)$, the Hamiltonian in (23) contains long-range two-body interaction terms between the Fermions[4]

$$\mathcal{H}_{\rm int} = -\frac{1}{4}{\rm Tr}\left[\left(\frac{\partial}{\partial\Phi}\right)U'(\Phi)\right] = -\frac{1}{4}\sum_{i,j}\frac{U'(x_i)-U'(x_j)}{x_i-x_j}. \tag{26}$$

Thus, a one-dimensional collection of eigenvalues may be considered as an interacting Fermi gas.

The eigenstates of $\mathcal{H}$ have definite $U(N)$ quantum numbers and thus fall into definite $U(N)$ representations. Because of this symmetry it is useful to introduce matrix polar coordinates $\Phi = \Omega X \Omega^\dagger$, where $\Omega \in U(N)$ and X is a real diagonal matrix whose entries, x_1, $x_2 \ldots, x_N$, are the eigenvalues of Φ. The transformation from Cartesian matrix coordinates Φ_{ij} to polar matrix coordinates Ω_{ij}, and $X_{ij} = x_i\delta_{ij}$ is associated with a Jacobian, which is proportional to $\Delta^2(x_i)$. Under this transformation $\mathcal{H}$ and Ψ_0 transform as

$$\begin{aligned} \mathcal{H} &\rightarrow \Delta\mathcal{H}\Delta^{-1}, \\ \Psi_0 &\rightarrow \Delta\Psi_0. \end{aligned} \tag{27}$$

This implies that the transformed self-adjoint Hamiltonian is symmetric and that the transformed ground state is totally anti-symmetric under the interchange of any two of the eigenvalues x_i. These symmetry properties remain valid for any $U(N)$ singlet eigenstate of $\mathcal{H}$. Therefore, the singlet sector of (23) is equivalent to a one-dimensional Fermi gas with a fixed number N of particles [18], where the eigenvalues x_i are considered as the coordinates of the N Fermi particles.

Let us consider the class of singlet states of the form

$$\Psi_V(\Phi) = \exp[-\frac{1}{2}{\rm Tr}V(\Phi)], \tag{28}$$

[4]Note that if U is a polynomial of degree less than or equal to three, there are no two-body interactions in (26).

where Ψ_V is assumed to be normalizable. In matrix polar coordinates this becomes

$$\Psi_V(x_i) = \mathcal{N}\mathrm{det}_{i,j}\left(x_j^{i-1}\right)\exp\left[-\frac{1}{2}\sum_{k=1}^{N} V(x_k)\right] \quad (1 \leq i, j \leq N), \tag{29}$$

where we have used the identity $\Delta(x_j) = \mathrm{det}_{i,j}\,(x_j^{i-1})$ and $\mathcal{N}$ is a normalization constant.

It now becomes clear how this discussion of a Fermi gas relates to the theory of polynomials [3]. First, note that

$$\mathrm{det}_{i,j}\left(x_j^{i-1}\right) = \mathrm{det}_{i,j}\left[P_{i-1}(x_j)\right] \quad (1 \leq i, j \leq N) \tag{30}$$

for any set of *monic* polynomials $P_i(x) = x^i + (\text{lower powers of } x)$. Thus, the wave function (29) is a Slater determinant [19]:

$$\Psi_V(x_i) = \frac{1}{\sqrt{N!}}\mathrm{det}_{i,j}\left[v_{i-1}(x_j)\right], \tag{31}$$

where $v_i(x) = \nu_i P_i(x)e^{-V(x)/2}$ are interpreted as normalized single particle states in the N fermion wave function (31), which will be used as a trial wave function in a variational (Hartree-Fock) calculation of the ground state of the Fermi gas and $\mathcal{N}$ is a normalization constant.

Therefore, these functions must satisfy the orthonormality condition[19]

$$\int_{-\infty}^{\infty} dx\, e^{-V(x)} v_i(x)v_j(x) = \delta_{ij}, \tag{32}$$

which implies that the polynomials $\{P_i(x)\}$ in (30) are the set of polynomials orthogonal on the real line with respect to the weight $e^{-V(x)}$. In particular, for $V(x) = U(x)$, we have $\Psi_V(x) = \Psi_U(x) \equiv \Psi_0(x)$ and $\{v_i(x)\}$ are identical to the orthonormal functions $\{u_i(x)\}$ defined in (6).

Consider the expectation value of the Hamiltonian $\mathcal{H}$ in (23) in the normalizable state of (28):

$$\mathcal{E}_0(V) = \frac{1}{\mathcal{Z}_V}\int d^{N^2}\Phi\, e^{-\frac{1}{2}\mathrm{Tr}V}\mathcal{H}e^{-\frac{1}{2}\mathrm{Tr}V}, \tag{33}$$

where

$$\mathcal{Z}_V = \int d^{N^2}\Phi\, e^{-\mathrm{Tr}V} \;=\; \langle\Psi_V|\Psi_V\rangle < \infty. \tag{34}$$

Clearly,

$$\mathcal{E}_0(V) \geq 0 \tag{35}$$

for any normalizable Ψ_V.

Since we have

$$\mathcal{E}_0(U) = 0, \quad \mathcal{Z}_U < \infty, \tag{36}$$

we expect $\Psi_U \equiv \Psi_0$ to be an absolute quadratic minimum for $\mathcal{E}_0(V)$ in the space of functions of the form given by (28) and (34). Indeed, for V infinitesimally different from U,

$$V = U + \delta U, \tag{37}$$

we find that

$$\mathcal{E}_0(U+\delta U) = \langle\Psi_U|\frac{1}{8}\mathrm{Tr}(\delta U')^2|\Psi_U\rangle/\mathcal{Z}_U + \mathcal{O}(\delta U)^3 \geq 0. \tag{38}$$

Expressing (38) in matrix polar coordinates and using the Slater determinant representation (31) for Ψ_V (and for Ψ_U), we find, after some trivial manipulations, that

$$\begin{aligned}\mathcal{E}_0(U+\delta U) &= \sum_{i=0}^{N-1}\int dx v_i^*(x)\left[-\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{4}U''(x) + \frac{1}{8}U'^2(x) + \mu_i^{(N)}\right]v_i(x) \\ &-\frac{1}{4}\Sigma'_{i,j}\int dxdy\, v_i^*(x)v_j^*(y)\frac{U'(x)-U'(y)}{x-y}[v_i(x)v_j(y) - v_j(x)v_i(y)] - \sum_{i=0}^{N-1}\mu_i^{(N)} \\ &= \frac{1}{8}\sum_{i=0}^{N-1}\int dx|u_i(x)|^2(\delta U'(x))^2 + \mathcal{O}(\delta U)^3.\end{aligned} \tag{39}$$

Here, the $\{\mu_i^{(N)}\}_{i=0}^{N-1}$ are a set of N Lagrange multipliers (also known as chemical potentials) that will enforce the unit normalization condition on the $\{v_i\}$ in the variational calculation.[5]

[5] It is unnecessary to include in (39) Lagrange multipliers to enforce *all* orthonormality conditions (32) because the $\{v_i\}$ which result from the variational calculation turn out *a posteriori* to be orthonormal[19].

Taking the variational derivative of (39) with respect to $v_i^*(x)$, at $V = U$, we find that

$$\delta\mathcal{E}_0(U+\delta V)/\delta v_i^*(x)|_{\delta U=0} = \{[-\frac{1}{2}\partial_x^2 - \frac{1}{4}U'' + \frac{1}{8}U'^2 + \mu_i^{(N)}]v_i(x)$$
$$-\frac{1}{2}\Sigma'_j \int dy v_j^*(y)\frac{U'(x)-U'(y)}{x-y}[v_i(x)v_j(y) - v_j(x)v_i(y)]\}|_{v_i=u_i}$$
$$= \frac{1}{4}\sum_{i=1}^{N}\int dy|u_i(y)|^2\delta U'(y)\frac{\delta U'(y)}{\delta v_i^*(x)}|_{\delta U=0} \equiv 0. \tag{40}$$

This expression must vanish because

$$K_i(x,y) = \frac{\delta U'(x)}{\delta v_i(y)} \tag{41}$$

must be a regular kernel in function space for V infinitesimally close to U because $V(x)$ defines the $v_i(x)$ uniquely, and vice-versa $[V(x) = -2\log\left(\frac{v_0(x)}{\mathcal{P}_0}\right)]$. At the point $V(x) = U(x)$ in function space the set of functions $\{v_i(x)\}$ coincides with the set of functions $\{u_i(x)\}$ defined in (6), where the $\{\mathcal{P}_i(x)\}$ are the set of polynomials orthonormal with respect to the weight $e^{-U(x)}$ on the real line. Moreover, uniqueness of the ground state of the quantum-mechanical system defined by (23) implies that (6) is the *unique* solution to (1). This completes our proof of the assertion made in Eqs. (1) and (6).

We conclude this section by proving that the first ν orthonormal polynomials, $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_\nu$, associated with the weight $w^2(x) = e^{-U(x)}$ satisfy the useful sum-rule

$$\sum_{m,n=0}^{\nu-1}\int_{-\infty}^{\infty}dx\, w^2(x)\int_{-\infty}^{\infty}dy\, w^2(y)\frac{U'(x)-U'(y)}{x-y}\mathcal{P}_n(x)\mathcal{P}_m(y)\left[\mathcal{P}_n(x)\mathcal{P}_m(y) - \mathcal{P}_m(x)\mathcal{P}_n(y)\right]$$
$$= 2\sum_{n=0}^{\nu-1}\int_{-\infty}^{\infty}dx\, w^2(x)[\mathcal{P}_n'(x)]^2. \tag{42}$$

To prove (42) we let Φ be a $\nu\times\nu$ Hermitian matrix and define

$$S_\nu = \frac{\int d^{\nu^2}\Phi e^{-\mathrm{Tr}U(\Phi)}\mathrm{Tr}\left[\left(\frac{\partial}{\partial\Phi}\right)U'(\Phi)\right]}{\int d^{\nu^2}\Phi e^{-\mathrm{Tr}U(\Phi)}} \tag{43}$$

Integrating by parts over Φ we obtain straight forwardly

$$S_\nu = \frac{\int d^{\nu^2}\Phi e^{-\mathrm{Tr}U(\Phi)}\mathrm{Tr}\left[U'^2(\Phi)\right]}{\int d^{\nu^2}\Phi e^{-\mathrm{Tr}U(\Phi)}}. \tag{44}$$

In terms of the Slater determinant (31) (with $V = U$) the latter equation becomes[19]

$$S_\nu = \sum_{n=0}^{\nu-1} \int_{-\infty}^{\infty} dx\ w^2(x) \mathcal{P}_n^2(x) U'^2(x). \tag{45}$$

On the other hand, using the identity

$$\mathrm{Tr}\left[\left(\frac{\partial}{\partial\Phi}\right) U'(\Phi)\right] = \sum_{m,n=1}^{\nu} \frac{U'(x_m) - U'(x_n)}{x_m - x_n} = \sum_{m=1}^{\nu} U''(x_m) + \sum_{m\neq n=1} \frac{U'(x_m) - U'(x_n)}{x_m - x_n} \tag{46}$$

and the same Slater determinant as above, we obtain[19] an alternative expression for S_ν as

$$\begin{aligned} S_\nu = &\\ &\sum_{m,n=0}^{\nu-1} \int_{-\infty}^{\infty} dx w^2(x) \int_{-\infty}^{\infty} dy w^2(y) \frac{U'(x) - U'(y)}{x - y} \mathcal{P}_n(x) \mathcal{P}_m(y) \left[\mathcal{P}_n(x)\mathcal{P}_m(y) - \mathcal{P}_m(x)\mathcal{P}_n(y)\right] \\ &\quad + \sum_{n=0}^{\nu-1} \int_{-\infty}^{\infty} dx w^2(x) \mathcal{P}_n^2(x) U''(x). \end{aligned} \tag{47}$$

Equating (45) and (47), using the identity $e^{-U(x)}[U'^2(x) - U''(x)] = \frac{d^2}{dx^2} e^{-U(x)}$ and the orthonormality condition (5), we obtain (42) as required.

3 Eigenvalues of the Integro-Differential Equation

In the previous section we have proved that the set of functions in (6) is the unique solution of the system of integro-differential equations in (1). The eigenvalues $\{\mu_i^{(N)}\}$ of (1) played absolutely no role in that proof because we already knew the exact form (24) of the Fermi gas ground state and used it to show that (1) and (6) hold. However, these eigenvalues are an indispensable ingredient of the system (1) because they are Lagrange multipliers that enforce the orthonormality conditions (32). They therefore encode important information about the orthonormal polynomials (6) in much the same way such information is encoded by the set of moments of the weight $e^{-U(x)}$ or by the set of coefficients of the generic three-term recursion relations among the polynomials $\mathcal{P}_i$ [20, 21, 1, 3].[6] Since the polynomials $\mathcal{P}_i$ are known once the weight $e^{-U(x)}$ is given, we can use (1) to determine the eigenvalues $\{\mu_i^{(N)}\}$.

Multiplying (1) by $u_i^*(x)$ from (6) and integrating over x we find, using (2), that the eigenvalues are given by

$$\mu_n^{(N)} = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, w^2(x)\,(\mathcal{P}_n')^2 + \frac{1}{2}\sum_{m=0}^{N-1} K_{nm} \quad (0 \le n \le N-1). \tag{48}$$

Here, as before, $w^2(x) = e^{-U(x)}$ and

$$\begin{aligned} K_{nm} &= K_{mn} \\ &= \int_{-\infty}^{\infty} dx\, w^2(x) \int_{-\infty}^{\infty} dy\, w^2(y) \frac{U'(x) - U'(y)}{x-y} \\ &\qquad \times \mathcal{P}_n(x)\,\mathcal{P}_m(y)\left[\mathcal{P}_n(x)\,\mathcal{P}_m(y) - \mathcal{P}_n(y)\,\mathcal{P}_m(x)\right] \\ &\equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \mathcal{K}_{nm}(x,y), \end{aligned} \tag{49}$$

where we define

$$\mathcal{K}_{nm}(x,y) \equiv w^2(x) w^2(y)\,(\partial_x - \partial_y)\,\frac{\mathcal{P}_n(x)\,\mathcal{P}_m(y)\left[\mathcal{P}_n(x)\,\mathcal{P}_m(y) - \mathcal{P}_n(y)\,\mathcal{P}_m(x)\right]}{x-y}. \tag{50}$$

[6] Note, however, that the latter two sets of coefficients involve *all* polynomials, while the eigenvalues $\{\mu_i^{(N)}\}$ do depend on the number N of polynomials used.

It is obvious from (49) that the symmetric matrix K_{nm} has vanishing diagonal elements

$$K_{nn} = 0. \tag{51}$$

The sum-rule (42) we have found in the previous section now becomes

$$\sum_{n,\, m=0}^{N-1} K_{nm} = 2 \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} dx\, w^2(x) \left(\mathcal{P}_n'\right)^2 . \tag{52}$$

Using (49)-(52) and the symmetry of the matrix K we obtain after some algebra yet another sum rule

$$\sum_{m=0}^{N-1} K_{nm} = \sum_{k=n}^{N-1} \int_{-\infty}^{\infty} dx\, w^2(x) \left(\mathcal{P}_k'\right)^2 - \sum_{k=n+1}^{N-1} \sum_{l=0}^{N-1} K_{kl} + \frac{1}{2} \sum_{k,l=n}^{N-1} K_{kl} \quad (n \le N-2). \tag{53}$$

From (48) and (53) we infer the following recursion relation satisfied by the eigenvalues

$$\mu_n^{(N)} = - \sum_{k=n+1}^{N-1} \mu_k^{(N)} + \frac{1}{4} \sum_{k,l=n}^{N-1} K_{kl} \quad (n \le N-2).$$

The symmetry of the matrix K allows us to restrict the double sum on the right side of the last equation to the upper triangular part of K; the recursion relation then becomes

$$\mu_n^{(N)} = - \sum_{k=n+1}^{N-1} \mu_k^{(N)} + \frac{1}{2} \sum_{k=n}^{N-1} \sum_{p=1}^{N-k-1} K_{k,k+p} \quad (n \le N-2). \tag{54}$$

This recursion relation may be used to calculate $\mu_n^{(N)}$ for descending values of n holding N fixed. Similarly, (48) implies that along the N index direction there is a recursion relation of the form

$$\mu_n^{(N+1)} = \mu_n^{(N)} + \frac{1}{2} K_{n,N} \quad (0 \le n \le N-1). \tag{55}$$

The equations (55) and (54) are not independent and thus they cannot be combined to give a single self-contained recursion relation in terms of the eigenvalues $\mu_n^{(N)}$ alone. Nevertheless, we *can* calculate $\mu_n^{(N)}$ iteratively using the above formulas.

The initial condition for both (54) and (55), namely the value of $\mu_{N-1}^{(N)}$, must be calculated separately. We show below that

$$\mu_{N-1}^{(N)} \equiv 0 \tag{56}$$

for *any* even weight $w^2(x) = e^{-U(x)}$.

To prove this identity we note from (48) that

$$\mu_{N-1}^{(N)} = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, w^2(x) \left(\mathcal{P}'_{N-1}\right)^2 + \frac{1}{2}\sum_{m=0}^{N-1} K_{N-1,m}. \tag{57}$$

Rewriting the sum in the last equation as

$$\sum_{m=0}^{N-1} K_{N-1,m} = \left(\sum_{n,m=0}^{N-1} - \sum_{n,m=0}^{N-2}\right) K_{nm} - \sum_{m=0}^{N-1} K_{m,N-1}, \tag{58}$$

and using the sum-rule (52) and the symmetry of the matrix K, we find that

$$\sum_{m=0}^{N-1} K_{N-1,m} = 2\int_{-\infty}^{\infty} dx\, w^2(x) \left(\mathcal{P}'_{N-1}\right)^2 - \sum_{m=0}^{N-1} K_{N-1,m}. \tag{59}$$

Therefore,

$$\sum_{m=0}^{N-1} K_{N-1,m} = \int_{-\infty}^{\infty} dx\, w^2(x) \left(\mathcal{P}'_{N-1}\right)^2 \tag{60}$$

which upon substitution into (57), yields the result (56).

We now give explicit general formulas for the first few eigenvalues $\mu_n^{(N)}$. Our procedure is to use (49) and (50) to obtain a sequence of formulas for K_{nm} and thus for the eigenvalues. These formulas are obtained from the polynomial recursion relation [1, 3, 20, 21]

$$x\mathcal{P}_n(x) = \sqrt{R_{n+1}}\,\mathcal{P}_{n+1}(x) + \sqrt{R_n}\,\mathcal{P}_{n-1}(x), \tag{61}$$

where the recursion coefficients R_n are uniquely defined by the weight function $w^2(x)$. Note that this recursion relation generates normalized polynomials $\mathcal{P}_n(x)$:

$$\begin{aligned}
\mathcal{P}_0(x) &= 1,\\
\mathcal{P}_1(x) &= \frac{x}{\sqrt{R_1}},\\
\mathcal{P}_2(x) &= \frac{x^2 - R_1}{\sqrt{R_1 R_2}},\\
\mathcal{P}_3(x) &= \frac{x^3 - (R_1 + R_2)x}{\sqrt{R_1 R_2 R_3}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_4(x) &= \frac{x^4 - (R_1 + R_2 + R_3)x^2 + R_1 R_3}{\sqrt{R_1 R_2 R_3 R_4}}, \\
\mathcal{P}_5(x) &= \frac{x^5 - (R_1 + R_2 + R_3 + R_4)x^3 + (R_1 R_3 + R_1 R_4 + R_2 R_4)x}{\sqrt{R_1 R_2 R_3 R_4 R_5}}, \\
\mathcal{P}_6(x) &= \frac{1}{\sqrt{R_1 R_2 R_3 R_4 R_5 R_6}}[x^6 - (R_1 + R_2 + R_3 + R_4 + R_5)x^4 \\
&\quad +(R_1 R_3 + R_1 R_4 + R_1 R_5 + R_2 R_4 + R_2 R_5 + R_3 R_5)x^2 - R_1 R_3 R_5], \\
\mathcal{P}_7(x) &= \frac{1}{\sqrt{R_1 R_2 R_3 R_4 R_5 R_6}}[x^7 - (R_1 + R_2 + R_3 + R_4 + R_5 + R_6)x^5 + (R_1 R_3 \\
&\quad +R_1 R_4 + R_1 R_5 + R_1 R_6 + R_2 R_4 + R_2 R_5 + R_2 R_6 + R_3 R_5 + R_3 R_6 \\
&\quad +R_4 R_6)x^3 - (R_1 R_3 R_5 + R_1 R_3 R_6 + R_1 R_4 R_6 + R_2 R_4 R_6)x].
\end{aligned} \tag{62}$$

Inserting the above recursion relation into (50) repeatedly allows us to cancel the quantity $x - y$ in the denominator.[7] The orthonormality relation

$$\int_{-\infty}^{\infty} dx\, w^2(x) \mathcal{P}_n(x) \mathcal{P}_m(x) = \delta_{nm} \tag{63}$$

then simplifies the resulting expression dramatically. To present our final results we define a set of polynomials $\mathcal{Q}_n^{(N)}(x)$ that are conjugate to the polynomials $\mathcal{P}_n(x)$:

$$\begin{aligned}
\mathcal{Q}_0^{(N)}(x) &= 1, \\
\mathcal{Q}_1^{(N)}(x) &= \frac{x}{\sqrt{R_{N-1}}}, \\
\mathcal{Q}_2^{(N)}(x) &= \frac{x^2 - R_{N-1}}{\sqrt{R_{N-1} R_{N-2}}}, \\
\mathcal{Q}_3^{(N)}(x) &= \frac{x^3 - (R_{N-1} + R_{N-2})x}{\sqrt{R_{N-1} R_{N-2} R_{N-3}}}, \\
\mathcal{Q}_4^{(N)}(x) &= \frac{x^4 - (R_{N-1} + R_{N-2} + R_{N-3})x^2 + R_{N-1} R_{N-3}}{\sqrt{R_{N-1} R_{N-2} R_{N-3} R_{N-4}}}, \\
\mathcal{Q}_5^{(N)}(x) &= \frac{1}{\sqrt{R_{N-1} R_{N-2} R_{N-3} R_{N-4} R_{N-5}}}[x^5 - (R_{N-1} + R_{N-2} + R_{N-3} + R_{N-4})x^3 \\
&\quad +(R_{N-1} R_{N-3} + R_{N-1} R_{N-4} + R_{N-2} R_{N-4})x],
\end{aligned}$$

[7] For $|m - n| = 1$ in (49),(50) this iterative process yields the Darboux-Christoffel formula. When $|m - n| > 1$ one obtains generalizations thereof [20, 21].

$$
\begin{aligned}
\mathcal{Q}_6^{(N)}(x) &= \frac{1}{\sqrt{R_{N-1}R_{N-2}R_{N-3}R_{N-4}R_{N-5}R_{N-6}}}[x^6-(R_{N-1}+R_{N-2}+R_{N-3}+R_{N-4}\\
&\quad +R_{N-5})x^4+(R_{N-1}R_{N-3}+R_{N-1}R_{N-4}+R_{N-1}R_{N-5}+R_{N-2}R_{N-4}\\
&\quad +R_{N-2}R_{N-5}+R_{N-3}R_{N-5})x^2-R_{N-1}R_{N-3}R_{N-5}],\\
\mathcal{Q}_7^{(N)}(x) &= \frac{1}{\sqrt{R_{N-1}R_{N-2}R_{N-3}R_{N-4}R_{N-5}R_{N-6}}}[x^7-(R_{N-1}+R_{N-2}+R_{N-3}+R_{N-4}\\
&\quad +R_{N-5}+R_{N-6})x^5+(R_{N-1}R_{N-3}+R_{N-1}R_{N-4}+R_{N-1}R_{N-5}\\
&\quad +R_{N-1}R_{N-6}+R_{N-2}R_{N-4}+R_{N-2}R_{N-5}+R_{N-2}R_{N-6}+R_{N-3}R_{N-5}\\
&\quad +R_{N-3}R_{N-6}+R_{N-4}R_{N-6})x^3-(R_{N-1}R_{N-3}R_{N-5}\\
&\quad +R_{N-1}R_{N-3}R_{N-6}+R_{N-1}R_{N-4}R_{N-6}+R_{N-2}R_{N-4}R_{N-6})x], \qquad (64)
\end{aligned}
$$

where we have replaced R_n in (62) by R_{N-n}. For each value of N this set of polynomials is orthonormal as can be seen from the fact that they satisfy a three-term recursion relation whose recursion coefficients are R_{N-n}.

We then obtain the formula

$$
K_{n,n-k} = \frac{1}{\sqrt{R_n}}\int_{-\infty}^{\infty} dx\, w^2(x)\mathcal{P}_n'(x)\mathcal{Q}_{k-1}^{(n)}(x)\mathcal{P}_{n-k}(x). \qquad (65)
$$

Once we have determined the matrix K_{nm} we can then use (55) to construct a formula for the eigenvalues $\mu_n^{(N)}$:

$$
\mu_n^{(N)} = \frac{1}{2}\sum_{j=1}^{N-n-1} K_{n,N-j} \quad (n \le N-1). \qquad (66)
$$

We have been able to find closed-form expressions for some of the K_{nm}:

$$
\begin{aligned}
K_{n,n-1} &= \frac{n}{R_n} \quad (n\ge 1),\\
K_{n,n-2} &= \frac{2}{R_nR_{n-1}}\sum_{k=1}^{n-1} R_k \quad (n\ge 2),\\
K_{n,n-3} &= \frac{1}{R_nR_{n-1}R_{n-2}}\sum_{k=1}^{n-2}[R_{n-1}^2+2R_k^2+4R_kR_{k+1}-4R_{n-1}R_k] \quad (n\ge 3),
\end{aligned}
$$

$$
\begin{aligned}
K_{n,n-4} &= \frac{2}{R_n R_{n-1} R_{n-2} R_{n-3}} \sum_{k=1}^{n-3} [(R_{n-1}+R_{n-2})^2 R_k - 2(R_{n-1}+R_{n-2})(R_k^2 + 2R_k R_{k+1}) \\
&\quad + 3R_k R_{k+1} R_{k+2} + R_k^3 + 3R_k^2 R_{k+1} + 3R_k R_{k+1}^2] \quad (n \geq 4), \qquad (67)
\end{aligned}
$$

and so on. These formulas become increasingly complicated as one moves further away from the diagonal of the matrix K.

From (66) and (67) we obtain the following formulas for the first few eigenvalues:

$$
\begin{aligned}
\mu_{N-1}^{(N)} &= 0 \quad (N \geq 1), \\
\mu_{N-2}^{(N)} &= \frac{N-1}{2R_{N-1}} \quad (N \geq 2), \\
\mu_{N-3}^{(N)} &= \frac{1}{2R_{N-1}R_{N-2}}[2(R_1 + \ldots + R_{N-2}) + (N-2)R_{N-1}] \quad (N \geq 3), \qquad (68)
\end{aligned}
$$

and so on. Again, these formulas become more complicated as the difference between the two indices of μ grows.

4 Eigenvalues of the Hahn-Meixner Polynomials

In the previous section our discussion was completely general; it applies to the case of an even weight function $w^2(x)$ on the whole real line. To illustrate the results of the previous section we now focus on a special class of polynomials known as the Hahn-Meixner polynomials. In the context of the discussion above, these polynomials are single-particle states from which one constructs the ground state of a one-dimensional Fermi gas.

The Hahn-Meixner polynomials [6, 7, 8, 9, 10, 11] are a four-parameter family of polynomials $\mathcal{P}_n(x)$ orthogonal on the whole real line ($-\infty < x < \infty$). These polynomials are expressible in terms of a generalized hypergeometric function:

$$\mathcal{P}_n(x) = N_n i^n \, {}_3F_2(-n, n+a+b+c+d-1, a-ix; a+c, a+d; 1), \tag{69}$$

where a, b, c, and d are parameters that may take on any complex values and the normalization N_n is given by

$$N_n = \frac{[(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)\Gamma(n+a+c)\Gamma(n+a+d)]^{1/2}}{\Gamma(a+c)\Gamma(a+d)n!\,[\Gamma(n+b+c)\Gamma(n+b+d)]^{1/2}}. \tag{70}$$

These polynomials are orthonormal on the real line

$$\int_{-\infty}^{\infty} dx\, w^2(x)\mathcal{P}_m(x)\mathcal{P}_n(x) = \delta_{mn}, \tag{71}$$

where the weight $w^2(x)$ is given by

$$w^2(x) = \frac{1}{2\pi}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix). \tag{72}$$

Hahn-Meixner polynomials appear often in mathematical physics; for example, they play a major role in the representation theory of the Lorentz and rotation groups [11, 12]. Recently, it has been shown that a special case of the Hahn-Meixner polynomials plays a crucial role in discrete-time quantum mechanics [13], the solution of operator differential

equations [14], the operator ordering problem in quantum mechanics [13], and in the theory of Weyl-ordered (symmetric) operators of the Heisenberg algebra [10]. Here we restrict our attention to the special case of Hahn-Meixner polynomials that appear in the latter applications. These particular polynomials are characterized by the parameter values

$$a = c = \frac{1}{4} \quad \text{and} \quad b = d = \frac{3}{4} \tag{73}$$

in (69).

This special case of Hahn-Meixner polynomials may be obtained in a natural way from the Heisenberg algebra as follows. We introduce a complete operator basis having a simple commutator algebra. This Hermitian basis $T_{m,n}$ is the sum over all possible orderings of m operators p and n operators q. For example,

$$\begin{aligned}
T_{1,1} &= pq + qp, \\
T_{0,1} &= q, \\
T_{2,1} &= p^2q + pqp + qp^2, \\
T_{2,2} &= (p^2q^2 + q^2p^2 + pq^2p + qp^2q + qpqp + pqpq).
\end{aligned} \tag{74}$$

These symmetrized operator products $T_{m,n}$ can be rewritten as Weyl-ordered sums in the operators p and q with the terms in the sums weighted by binomial coefficients [15, 16]:

$$T_{m,n} = \frac{(m+n)!}{2^n m!\, n!} \sum_{k=0}^{n} \binom{n}{k} q^k p^m q^{(n-k)}. \tag{75}$$

These basis element operators $T_{m,n}$ have particularly simple commutation properties:

$$\begin{aligned}
[q, T_{m,n}] &= i(m+n)T_{m-1,n}, \\
[p, T_{m,n}] &= -i(m+n)T_{m,n-1}, \\
\{q, T_{m,n}\}_+ &= \frac{2(n+1)}{m+n+1}T_{m,n+1}, \\
\{p, T_{m,n}\}_+ &= \frac{2(m+1)}{m+n+1}T_{m+1,n}.
\end{aligned} \tag{76}$$

Thus, commuting with q and p has the effect of a lowering operator and anticommuting with q and p has the effect of a raising operator in the appropriate index. From the four basic algebraic relations (76) one can deduce a general formula for the commutation and anticommutation relations between $T_{m,n}$ and $T_{r,s}$[17].

The connection between the totally symmetrized operators $T_{m,n}$ and the special case of the Hahn-Meixner polynomials in (69) and (73) is

$$\begin{aligned} T_{n,n} &= \frac{1}{(2n-1)!!}\mathcal{P}_n(T_{1,1}), \\ T_{n,n+k} &= \frac{(2n+k)!}{(n+k)!2^{n+1}}\{q^k, \mathcal{P}_n(T_{1,1})\}_+. \end{aligned} \tag{77}$$

We list below the properties of the special Hahn-Meixner polynomials:

(i) The $\mathcal{P}_n(x)$ satisfy the recurrence relation

$$n\mathcal{P}_n(x) = x\mathcal{P}_{n-1}(x) - (n-1)\mathcal{P}_{n-2}(x). \tag{78}$$

By comparing this special case with the general recursion relation in (61) we identify

$$R_n = n^2. \tag{79}$$

(ii) The $\mathcal{P}_n(x)$ are orthonormal on the interval $(-\infty, \infty)$ with respect to the weight function

$$e^{-U(x)} = w^2(x) = \frac{1}{2\cosh(\pi x/2)}. \tag{80}$$

The moments of this weight function are Euler numbers:

$$\int_{-\infty}^{\infty} dx\, w^2(x) x^{2n} = |E_{2n}|, \tag{81}$$

where $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, and so on.

(iii) The $\mathcal{P}_n(x)$ have a simple generating function

$$\sum_{n=0}^{\infty} t^n \mathcal{P}_n(x) = \frac{\exp(x \arctan t)}{\sqrt{1+t^2}}.$$

(iv) The first few $\mathcal{P}_n(x)$ are

$$\begin{aligned}
\mathcal{P}_0(x) &= 1, \\
\mathcal{P}_1(x) &= x, \\
\mathcal{P}_2(x) &= \frac{1}{2}\left(x^2 - \frac{1}{2}\right), \\
\mathcal{P}_3(x) &= \frac{1}{6}\left(x^3 - 2x\right), \\
\mathcal{P}_4(x) &= \frac{1}{24}\left(x^4 - 5x^2 + \frac{3}{2}\right), \\
\mathcal{P}_5(x) &= \frac{1}{5!}\left(x^5 - 10x^3 + \frac{23}{2}x\right), \\
\mathcal{P}_6(x) &= \frac{1}{6!}\left(x^6 - \frac{35}{2}x^4 + 49x^2 - \frac{45}{4}\right), \\
\mathcal{P}_7(x) &= \frac{1}{7!}\left(x^7 - 28x^5 + 154x^3 - 132x\right), \\
\mathcal{P}_8(x) &= \frac{1}{8!}\left(x^8 - 42x^6 + 399x^4 - 818x^2 + \frac{315}{2}\right).
\end{aligned}$$

Using these polynomials in (65) we obtain the matrix K_{mn}, whose first few entries are listed in Table 1.

	m=0	m=1	m=2	m=3	m=4	m=5	m=6	m=7
$n=0$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{13}{36}$	$\frac{7}{20}$	$\frac{17}{60}$	$\frac{1727}{6300}$
$n=1$	1	0	$\frac{1}{2}$	$\frac{5}{18}$	$\frac{11}{36}$	$\frac{41}{180}$	$\frac{209}{900}$	$\frac{1201}{6300}$
$n=2$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{7}{36}$	$\frac{67}{300}$	$\frac{17}{100}$	$\frac{1121}{6300}$
$n=3$	$\frac{1}{2}$	$\frac{5}{18}$	$\frac{1}{3}$	0	$\frac{1}{4}$	$\frac{3}{20}$	$\frac{53}{300}$	$\frac{6017}{44100}$
$n=4$	$\frac{13}{36}$	$\frac{11}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	0	$\frac{1}{5}$	$\frac{11}{90}$	$\frac{461}{3150}$
$n=5$	$\frac{7}{20}$	$\frac{41}{180}$	$\frac{67}{300}$	$\frac{3}{20}$	$\frac{1}{5}$	0	$\frac{1}{6}$	$\frac{13}{126}$
$n=6$	$\frac{17}{60}$	$\frac{209}{900}$	$\frac{17}{100}$	$\frac{53}{300}$	$\frac{11}{90}$	$\frac{1}{6}$	0	$\frac{1}{7}$
$n=7$	$\frac{1727}{6300}$	$\frac{1201}{6300}$	$\frac{1121}{6300}$	$\frac{6017}{44100}$	$\frac{461}{3150}$	$\frac{13}{126}$	$\frac{1}{7}$	0

Table 1 The first few entries of the symmetric matrix K_{mn} associated with the Hahn-Meixner polynomials. The entries in this matrix were computed from (65).

Applying (66) to the entries in Table 1 we can compute the eigenvalues $\mu_n^{(N)}$ associated

with the Hahn-Meixner polynomials. The first few eigenvalues are listed in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10
$N=1$	0										
$N=2$	$\frac{1}{2}$	0									
$N=3$	$\frac{3}{4}$	$\frac{1}{4}$	0								
$N=4$	1	$\frac{7}{18}$	$\frac{1}{6}$	0							
$N=5$	$\frac{85}{72}$	$\frac{13}{24}$	$\frac{19}{72}$	$\frac{1}{8}$	0						
$N=6$	$\frac{61}{45}$	$\frac{59}{90}$	$\frac{169}{450}$	$\frac{1}{5}$	$\frac{1}{10}$	0					
$N=7$	$\frac{539}{360}$	$\frac{463}{600}$	$\frac{829}{1800}$	$\frac{173}{600}$	$\frac{29}{180}$	$\frac{1}{12}$	0				
$N=8$	$\frac{286}{175}$	$\frac{2731}{3150}$	$\frac{577}{1050}$	$\frac{3931}{11025}$	$\frac{41}{175}$	$\frac{17}{126}$	$\frac{1}{14}$	0			
$N=9$	$\frac{14713}{8400}$	$\frac{8083}{8400}$	$\frac{7331}{11760}$	$\frac{75703}{176400}$	$\frac{51403}{176400}$	$\frac{199}{1008}$	$\frac{13}{112}$	$\frac{1}{16}$	0		
$N=10$	$\frac{3917}{2100}$	$\frac{46057}{44100}$	$\frac{489}{700}$	$\frac{21607}{44100}$	$\frac{20009}{56700}$	$\frac{145}{588}$	$\frac{43}{252}$	$\frac{11}{108}$	$\frac{1}{18}$	0	
$N=11$	$\frac{346753}{176400}$	$\frac{22067}{19600}$	$\frac{134737}{176400}$	$\frac{877307}{1587600}$	$\frac{214189}{529200}$	$\frac{9257}{317520}$	$\frac{22633}{105840}$	$\frac{541}{3600}$	$\frac{49}{540}$	$\frac{1}{20}$	0

Table 2 Table of the eigenvalues $\mu_n^{(N)}$ associated with the Hahn-Meixner polynomials. These eigenvalues are obtained from the entries in Table 1 using (66).

For the case of the Hahn-Meixner polynomials the general formulas in (68) become

$$\begin{aligned}
\mu_{N-1}^{(N+1)} &= \frac{1}{2N}, \\
\mu_{N-2}^{(N+1)} &= \frac{5N-1}{6(N-1)N}, \\
\mu_{N-3}^{(N+1)} &= \frac{19N^2-28N+3}{15(N-2)(N-1)N}, \\
\mu_{N-4}^{(N+1)} &= \frac{171N^3-633N^2+551N-45}{105(N-3)(N-2)(N-1)N}, \\
\mu_{N-5}^{(N+1)} &= \frac{1289N^4-8980N^3+19351N^2-13052N+840}{630(N-4)(N-3)(N-2)(N-1)N}, \\
\mu_{N-6}^{(N+1)} &= \frac{16757N^5-187721N^4+738847N^3-1201015N^2+689652N-37800}{6930(N-5)(N-4)(N-3)(N-2)(N-1)N}, \\
\mu_{N-7}^{(N+1)} &= [2(63705N^6-1048800N^5+6531490N^4-19071030N^3+25902989N^2 \\
&\qquad -13249854N+623700)]
\end{aligned}$$

$$/[45045(N-6)(N-5)(N-4)(N-3)(N-2)(N-1)N],$$

$$\begin{aligned}
\mu_{N-8}^{(N+1)} &= [2(72199N^7 - 1639133N^6 + 14772758N^5 - 67196100N^4 + 161413785N^3] \\
&\quad -192719779N^2 + 90335010N - 3783780)] \\
&\quad /[45045(N-7)(N-6)(N-5)(N-4)(N-3)(N-2)(N-1)N], \quad (82)
\end{aligned}$$

and so on.

We have discovered that these formulas simplify considerably if they are rewritten in the form of partial fractions:

$$\begin{aligned}
2\cdot 1!!\mu_{N-1}^{(N+1)} &= \frac{1}{N}, \\
2\cdot 3!!\mu_{N-2}^{(N+1)} &= \frac{1}{N} + \frac{4}{N-1}, \\
2\cdot 5!!\mu_{N-3}^{(N+1)} &= \frac{3}{N} + \frac{12}{N-1} + \frac{23}{N-2}, \\
2\cdot 7!!\mu_{N-4}^{(N+1)} &= \frac{15}{N} + \frac{44}{N-1} + \frac{107}{N-2} + \frac{176}{N-3}, \\
2\cdot 9!!\mu_{N-5}^{(N+1)} &= \frac{105}{N} + \frac{276}{N-1} + \frac{693}{N-2} + \frac{1104}{N-3} + \frac{1689}{N-4}, \\
2\cdot 11!!\mu_{N-6}^{(N+1)} &= \frac{945}{N} + \frac{2340}{N-1} + \frac{4773}{N-2} + \frac{9360}{N-3} + \frac{13329}{N-4} + \frac{19524}{N-5}, \\
2\cdot 13!!\mu_{N-7}^{(N+1)} &= \frac{10395}{N} + \frac{24780}{N-1} + \frac{46767}{N-2} + \frac{94512}{N-3} + \frac{138027}{N-4} + \frac{185772}{N-5} + \frac{264207}{N-6}, \\
2\cdot 15!!\mu_{N-8}^{(N+1)} &= \frac{135135}{N} + \frac{313740}{N-1} + \frac{566235}{N-2} + \frac{978480}{N-3} + \frac{1694655}{N-4} + \frac{2264940}{N-5} \\
&\quad + \frac{2944395}{N-6} + \frac{4098240}{N-7}.
\end{aligned} \quad (83)$$

Moreover, we have found that many of the coefficients can be expressed using very simple formulas. For example, the leftmost coefficient is a double factorial: $1 = (-1)!!$, $1 = 1!!$, $3 = 3!!$, $15 = 5!!$, $105 = 7!!$, $945 = 9!!$, $10395 = 11!!$, and $135135 = 13!!$. Also, the rightmost coefficient is the sum of inverse odd integers:

$$1 = 1!!\left(\frac{1}{1}\right),$$

$$
\begin{aligned}
4 &= 3!!\left(\frac{1}{1}+\frac{1}{3}\right),\\
23 &= 5!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}\right),\\
176 &= 7!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right),\\
1689 &= 9!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right),\\
19524 &= 11!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\right),\\
26247 &= 13!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right),\\
4098240 &= 15!!\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}\right).
\end{aligned}
\tag{84}
$$

Nevertheless, we have not yet discovered a general formula for the coefficients.

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